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Differential Constrains, Recursion Operators, and Logical Integrability

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Differential constraints compatible with the linearized equations of partial differential equations are examined. Recursion operators are obtained by integrating the differential constraints.

1. INTRODUCTION

The classical work of Lie on group-invariant solutions generalizes well-known methods for finding similarity solutions and traveling wave solutions (Olver, 1993). The essence of the method is that the group in question is a symmetry group of the original system of partial differential equations. Olver and Rosenau (1986, 1987) proposed a generalization of the nonclassical method of Bluman and Cole (1969). In their formulation, the original system of partial differential equations can be enlarged by appending differential constraints such that the resulting overdetermined system of partial differential equations satisfy a compatibility condition.

This work is about differential constraints compatible with the linearized equations of partial differential equations. The relation between differential constraints and recursion operators is examined. For equations in the form $q_t = P(q, q_x, q_{xx})$ and $q_t = P(q, q_x, q_{xx}, q_{xxx})$, recursion operators are obtained by integrating the compatible differential constraints. Results are compared with Fokas' (1980) generalized symmetry approach.

2. DIFFERENTIAL CONSTRAINTS COMPATIBLE WITH LINEARIZED EQUATIONS

We can describe the differential constraint method for evolutionary equations

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$$q_t = P(q, q_x, q_{xx}, \ldots) \tag{1}$$

in the following way. First we linearize the given differential equation. In other words, we replace q (and its derivatives) in (1) by $q + \epsilon\Psi$ and differentiate both sides of the resulting expression with respect to ϵ and take the limit $\epsilon \to 0$. We have

$$\Psi_t = D_P(\Psi) \tag{2}$$

where D_P is the Fréchet derivative (Olver, 1993). The equation above can also be written as

$$\Psi_t = \sum_{i=0}^{N} P_i \Psi_i = \sum_{i=0}^{N} \frac{\partial P}{\partial q_i} \Psi_i \tag{3}$$

where N is the order of the differential equation, $q_0 = q$, $q_1 = q_x$, $q_2 = q_{xx}$, $\Psi_0 = \Psi$, $\Psi_1 = \Psi_x$, $\Psi_2 = \Psi_{xx}$, and so on. In the symmetry approach (2) is the main equation, and Ψ is the symmetry of the differential equation and is a function of q_i.

The compatible differential constraint is

$$H\Psi = 0 \tag{4}$$

where H depends on q_i. Its order (highest derivative in H) is at least N; then (4) may be written as

$$\Psi_N = \sum_{i=0}^{(N-1)} A_i \Psi_i \tag{5}$$

where $A_0, A_1, \ldots, A_{N-1}$ are functions of q_i. Compatibility of (5) and (2)

$$\Psi_{N,t} - \Psi_{t,N} = 0 \tag{6}$$

using (2) and (5) gives,

$$\sum_{i=0}^{N-1} \Psi_i W_i = 0 \tag{7}$$

Letting

$$W_i = 0 \tag{8}$$

we obtain a system of partial differential equations among P_i, A_i, and their partial derivatives. The solution of this system will determine the differential constraint (4), which can be integrated to give

$$\Phi\Psi = 0 \tag{9}$$

where Φ is the recursion operator.

We consider differential equations of the form

$$q_t = f(q, q_x, q_{xx}) \tag{10}$$

The linearized form of the above equation can be given as

$$\Psi_t = \gamma\Psi_{xx} + \alpha\Psi_x + \beta\Psi \tag{11}$$

where α, β, γ are functions of q, q_x, q_{xx}. We consider a differential constraint having the same order as (10) in the form

$$\Psi_{xx} = A\Psi_x + B\Psi \tag{12}$$

The compatibility condition between (11) and (12) gives the following evolution equations:

$$A_t = \alpha_{xx} + \alpha_x A + \gamma_{xx}A + 2\gamma_x A_x + \gamma_x A^2 + 2\gamma_x B$$
$$+ 2\beta_x + A_{xx} + A_x\alpha + 2A_x\gamma A + 2B_x\gamma \tag{13}$$

$$B_t = 2\alpha_x B + \gamma_{xx}B + 2\gamma_x B_x + \gamma_x AB + \beta_{xx}$$
$$- \beta_x A + 2A_x\gamma B + B_{xx}\gamma + B_x\alpha \tag{14}$$

The solutions to the system (13)–(14) can be given with the linearized equation

$$\Psi_t = \eta\Psi_{xx} + \left[\frac{2\eta r_{qq}}{r_q}q_x + 2\eta r + \eta_1\right]\Psi_x$$
$$+ \left[\frac{\eta(r_{qqq}r_q - r_{qq}^2)}{r_q^2}q_x^2 + 2\eta r_q q_x\right]\Psi \tag{15}$$

and the compatible differential constraint

$$\Psi_{xx} = \left[\frac{q_{xx}}{q_x} - \frac{q_x r_{qq}}{r_q} - r\right]\Psi_x$$
$$+ \left[\frac{rq_{xx}}{q_x} - \frac{(r_{qqq}r_q - r_{qq}^2)q_x^2}{r_q^2} - 2r_q q_x\right]\Psi \tag{16}$$

Here η, η_1 are constants and r is an arbitrary function of q. The differential constraint (16) can be integrated to give

$$\Phi = D + \frac{r_{qq}q_x}{r_q} + r + q_x D^{-1} r_q \tag{17}$$

where $(D^{-1} f)(x) = \int_{-\infty}^{x} f(\zeta)\, d\zeta$. This is the recursion operator of Fokas (1980) and Ibragimov and Shabat (1980). The integrable equations can be given as

$$q_t = \eta q_{xx} + \frac{\eta r_{qq} q_x^2}{r_q} + (2\eta r + \eta_1) q_x \tag{18}$$

The above equations are classified by Fokas (1980), Svinolupov (1985), Ibragimov and Shabat (1980), and Olver (1994).

Next, we consider differential equations of the general form

$$q_t = P(q, q_x, q_{xx}, q_{xxx}) \tag{19}$$

The linearization of the above equation takes the form

$$\Psi_t = \alpha \Psi_{xxx} + \beta \Psi_{xx} + \gamma \Psi_x + \delta \Psi \tag{20}$$

where α, β, γ, δ are functions of q, q_x, q_{xx}, q_{xxx}. We consider the differential constraint having the same order as (19)

$$\Psi_{xxx} = A \Psi_{xx} + B \Psi_x + C \Psi \tag{21}$$

where A, B, and C are functions of q, q_x, q_{xx}, q_{xxx}. The compatibility determines algebraic equations

$$B = -\frac{2\alpha}{3\eta}, \qquad C = -\frac{1}{3\eta}(\alpha_x - 2\alpha A + 2\beta) \tag{22}$$

and evolution equations

$$\begin{aligned} A_t = {} & \alpha_x A + \beta_x + A_{xxx}\eta + 3A_{xx}\eta A \\ & + 3A_x^2 \eta + A_x \alpha + 3A_x \eta A^2 \end{aligned} \tag{23}$$

$$\begin{aligned} \alpha_t = {} & \frac{1}{2}(2\alpha_{xxx}\eta - 3\alpha_{xx}\eta A - 3\alpha_x A_x \eta + 2\alpha_x \alpha \\ & - 3\beta_{xx}\eta + 6\beta_x \eta A + 6A_x \beta \eta + 3\epsilon_t \eta) \end{aligned} \tag{24}$$

$$\begin{aligned} \beta_t = {} & \frac{1}{4}(3\alpha_{xxx}\eta A - 6\alpha_{xx}\eta A^2 - 3\alpha_x A_{xx}\eta \\ & - 12\alpha_x A_x \eta A + 4\alpha_x \beta + \beta_{xxx}\eta - 6\beta_{xx}\eta A + 4\beta_x \alpha \\ & + 12\beta_x \eta A^2 + 6A_{xx}\beta\eta + 24A_x \beta \eta A) \end{aligned} \tag{25}$$

where η is a constant and ϵ is a function of t.

The linearized form of the first class is

$$\Psi_t = \eta \Psi_{xxx} + \left[\frac{\rho_1}{2} q_x^2 + \rho\right] \Psi_x + \rho_q q_x \Psi \tag{26}$$

with compatible differential constraint

$$\Psi_{xxx} = \frac{q_{xx}}{q_x}\Psi_{xx} - \left[\frac{\rho_1}{3\eta}q_x^2 + \frac{2\rho}{3\eta}\right]\Psi_x - \frac{-2\rho q_{xx} + 3\rho_q q_x^2}{3\eta q_x}\Psi \tag{27}$$

where ρ_1 is constant and ρ is a function of q with the condition

$$\rho_{qqq} + \frac{4\rho_1}{3\eta}\rho_q = 0 \tag{28}$$

The recursion operator can be obtained by integration of (27),

$$\Phi = D^2 + \frac{2\rho}{3\eta} + \frac{\rho_1}{3\eta}q_x^2 - \frac{\rho_1}{3\nu}q_x D^{-1}q_{xx} + \frac{q_x}{\eta}D^{-1}\rho_q \tag{29}$$

The integrable equation is in the form

$$q_t = \eta q_{xxx} + \frac{\rho_1}{6}q_x^3 + \rho q_x \tag{30}$$

The second class is given by the linearized equation

$$\Psi_t = \eta\Psi_{xxx} + [\epsilon_1 q_x^2 + 2\epsilon_1\epsilon_2 q_x + 2\epsilon_3]\Psi \tag{31}$$

with compatible differential constraint

$$\Psi_{xxx} = \frac{q_{xx}}{q_x + \epsilon_2}\Psi_{xx} - \left[\frac{\epsilon_1}{3\eta}q_x^2 + \frac{2\epsilon_1\epsilon_2}{3\eta}q_x + \frac{2\epsilon_3}{3\eta}\right]\Psi_x - \frac{(\epsilon_2^2\epsilon_1 - 2\epsilon_3)q_{xx}}{3\eta(q_x + \epsilon_2)}\Psi \tag{32}$$

where ϵ_1, ϵ_2, ϵ_3 are constants. The recursion operator can be obtained by integration of (32),

$$\Phi = D^2 + \frac{2\epsilon_3}{3\eta} + \frac{\epsilon_2}{3\eta}q_x^2 + \frac{2\epsilon_1\epsilon_2}{3\eta}q_x - \frac{\epsilon_1}{3\eta}(q_x + \epsilon_2)D^{-1}q_{xxx} \tag{33}$$

The integrable equation is in the form

$$q_t = \eta q_{xxx} + \frac{\epsilon_1}{6}q_x^3 + \frac{\epsilon_1\epsilon_2}{2}q_x^2 + \epsilon_3 q_x \tag{34}$$

Equations (30) and (34) are classified by Fokas (1980) and Ibragimov and Shabat (1980) and recursion operators (29) and (33) are obtained by the integration of the differential constraints.

The third type of equation is given by the linearized equation

$$\Psi_t = \lambda_5\Psi_{xxx} + \frac{3}{2}\lambda_3\lambda_5\Psi_{xx} + \frac{3}{2}\lambda_4\lambda_5\Psi_x + \lambda_6\Psi \tag{35}$$

and differential constraint

$$\Psi_{xxx} = \left[\frac{q_{xxx} + \lambda_1 q_{xx} + \lambda_2 q_x}{q_{xx} + \lambda_1 q_x + \lambda_2 q} - \lambda_3\right]\Psi_{xx} + \left[\frac{\lambda_3(q_{xxx} + \lambda_1 q_{xx} + \lambda_2 q_x)}{q_{xx} + \lambda_1 q_x + \lambda_2 q} - \lambda_4\right]\Psi_x + \frac{\lambda_4(q_{xxx} + \lambda_1 q_{xx} + \lambda_2 q_x)}{q_{xx} + \lambda_1 q_x + \lambda_2 q}\Psi \tag{36}$$

The recursion operator is

$$\Phi = D^2 + \lambda_3 D + \lambda_1 \tag{37}$$

The integrable equation is

$$q_t = \lambda_5 q_{xxx} + \frac{3}{2}\lambda_3 \lambda_5 q_{xx} + \frac{3}{2}\lambda_4\lambda_5 q_x + \lambda_6 q \tag{38}$$

where λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 are constants. Rabelo and Tanenblat (1992) also obtained linear equation using the classification method of pseudospherical surfaces with Gaussian curvature (-1) [9].

3. CONCLUSION

This work has treated differential constraints compatible with the linearized equations of partial differential equations. For equations in the form $q_t = P(q, q_x, q_{xx})$ and $q_t = P(q, q_x, q_{xx}, q_{xxx})$, recursion operators were obtained by integrating the compatible differential constraints. We also mention that our classification is up to a change of variables. There are partial differential equations which are not included in the classification (10), but they can be obtained by a transformation [for example, equation (3.13) in Fakas (1987) by extended hodograph transformation]. There are equations which appear in the same class, and are related to each other by a Miura-type transformation.

The main idea in this work is to give the definition of logical integrability. A partial differential equation is logically integrable if its linearized equation is compatible with a differential constraint. This method is different from other approaches in the sense that it is only defined relative to a point q. This definition is also useful for classification purposes, because it can be applied to arbitrary equations (without polynomial restriction).

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